# UNIT – 1

# COMPLEX NUMBER

## INTRODUCTION

$$z = x + iy$$

$$x = R.P.\ \ of\ \ Z$$
$$y = I.P.\ \ of\ \ Z$$

$\bar{z} = x - iy$ is called **conjugate** of complex no. Also Real part of $z = x = \dfrac{z+\bar{z}}{2}$

Imaginary part of $z = y = \dfrac{z-\bar{z}}{2i}$

## POLAR FORM OF COMPLEX NUMBER:-

$x = r\cos\theta, \quad y = r\sin\theta$, (squaring and adding)

$x^2 + y^2 = r^2$ (Dividing above)

$$\tan\theta = \frac{y}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$r = \sqrt{x^2+y^2} = |z|$ is called **Modulus or absolute** value of z.

θ is called **argument or amplitude** of $x + iy$.

**Remark**:- Amplitude θ can take infinite values. The value of θ lies between – π and π is called **principal value** of the amplitude.

## PROPERTIES OF AMPLITUDE

I) $amp(z_1.z_2) = amp(z_1) + amp(z_2)$

II) $amp(z_1/z_2) = amp(z_1) - amp(z_2)$

## PROPERTIES OF MODULUS

I) $|z_1 + z_2| \le |z_1| + |z_2|$

II) $|z_1 - z_2| \ge |z_1| - |z_2|$

III) $|z|^2 = |\bar{z}| = z.\bar{z}$ i.e. $|x+iy|^2 = |x-iy|^2 = (x+iy)(x-iy) = x^2 + y^2$

IV) $amp(z) + amp(\bar{z}) = 0$

V) $|z_1.z_2| = |z_1|.|z_2|$

VI) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$

## DE – MOIVRE'S THEOREM

I) If n is any integer, +ve or –ve, then

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$$

II) If n is a fraction, +ve or –ve, then one of the values of $(\cos\theta + i\sin\theta)^{p/q}$ *is* $\cos\dfrac{p}{q}\theta + i\sin\dfrac{p}{q}\theta$

## REMARK:-

i) $(\cos\theta + i\sin\theta)^{-n} = \cos n\theta - i\sin n\theta$

ii) $(\cos\theta - i\sin\theta)^{n} = \cos n\theta - i\sin n\theta$

iii) $(\cos\theta - i\sin\theta)^{-n} = \cos n\theta + i\sin n\theta$

iv) $(\cos\theta + i\sin\theta)^{n} = \cos n\theta + i\sin n\theta$

v) $\dfrac{1}{\cos\theta + i\sin\theta} = \cos\theta - i\sin\theta$

vi) $(\sin\theta + i\cos\theta)^n \neq (\sin n\theta + i\cos n\theta)$

But $\left[\cos\left(\dfrac{\pi}{2}-\theta\right) + i\sin\left(\dfrac{\pi}{2}-\theta\right)\right]^n = \cos n\left(\dfrac{\pi}{2}-\theta\right) + i\sin n\left(\dfrac{\pi}{2}-\theta\right)$

vii) $(\cos\theta + i\sin\phi)^n \neq (\cos n\theta + i\sin n\phi)$ i.e. angle must me same while applying De – Moivre's Theorem.

viii) $cis\theta_1 . cis\theta_2 .............cis\theta_n = cis\left(\theta_1 + \theta_2 + .......\theta_n\right)$

OR

$e^{i\theta_1}.e^{i\theta_2}......e^{i\theta_n} = e^{i\left(\theta_1+\theta_2+.....\theta_n\right)}$

ix) $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

x) $\left(cism\theta\right)^n = cismn\theta = \left(cisn\theta\right)^m$

**ROOTS OF COMPLEX NUMBER**

There are q and only q distinct value of $\left(\cos\theta + i\sin\theta\right)^{\frac{1}{q}}$, where q being an +ve integer.

**PROCEDURE TO FIND OUT ROOTS**

I) Write $\left(\cos\theta + i\sin\theta\right)^{\frac{p}{q}} = \left(\cos p\theta + i\sin p\theta\right)^{1/q}$

$= \cos\left(\dfrac{p\theta + 2n\pi}{q}\right) + i\sin\left(\dfrac{p\theta + 2n\pi}{q}\right)$

Where n = 0, 1, 2, 3, 4, 5, ---- (q – 1).

Since $\cos\theta$ and $\sin\theta$ both general value of $2n\pi$ so we added $2n\pi$ in $\sin\theta$ and $\cos\theta$.

II) $1 = cis0, \quad -1 = cis\pi, \quad i = cis\,\pi/2, \quad -i = cis\left(-\pi/2\right)$

**Remark**:- Here it should be noted that while finding distinct values of $\left(cis\theta\right)^{p/q}$, $p/q$ always in its lowest terms.

e.g. $\left(\cos\theta + i\sin\theta\right)^{3/15}$ have only five roots, not 15 because $\dfrac{3}{15} = \dfrac{1}{5}$.

**CIRCULAR FUNCTION OF A COMPLEX VARIABLE**

i) $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

ii) $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

iii) $\tan z = \dfrac{e^{iz} - e^{-iz}}{e^{iz} + e^{-iz}}$

iv) $\cot z = \dfrac{e^{iz} + e^{-iz}}{e^{iz} - e^{-iz}}$

**HYPERBOLIC FUNCTION OF A COMPLEX VARIABLE**

i) $\sinh z = \frac{e^z - e^{-z}}{2}$ ii) $\cosh z = \frac{e^z + e^{-z}}{2}$ iii) $\tanh z = \frac{e^z - e^{-z}}{e^z + e^{-z}}$

iv) $\coth z = \frac{e^z + e^{-z}}{e^z - e^{-z}}$ v) $\sinh 0 = 0$ vi) $\cosh 0 = 1$

**FORMULAE OF HYPERBOLIC FUNCTION**

i) $\cosh^2 x - \sinh^2 x = 1$ ii) $\cosh^2 x + \sinh^2 x = \cosh 2x$

iii) $\sinh 2x = 2\sinh x \cosh x$ iv) $\cosh 2x = 2\cosh^2 x - 1 = 1 + 2\sinh^2 x$

v) $\sec h^2 x + \tanh^2 x = 1$ vi) $\coth^2 x - \cos ech^2 x = 1$

vii) $\sinh 3x = 3\sinh x + 4\sinh^3 x$ viii) $\cosh 3x = 4\cosh^3 x - 3\cosh x$

**RELATION BETWEEN HYPERBOLIC AND CIRCULAR FUNCTION**

i) $\sin ix = i\sinh x$ ii) $\cos ix = \cosh x$ iii) $\tan ix = i\tanh x$

iv) $\sinh ix = i\sin x$ v) $\cosh ix = \cos x$ vi) $\tanh ix = i\tan x$

**INVERSE HYPERBOLIC FUNCTION**

i) $\sinh^{-1} z = \log\left[z + \sqrt{z^2 + 1}\right]$ ii) $\cosh^{-1} z = \log\left[z + \sqrt{z^2 - 1}\right]$

iii) $\tanh^{-1} z = \frac{1}{2}\log\frac{1+z}{1-z}$

**LOGARITHMIC FUNCTION**

1. $Log(x+iy) = 2n\pi i + \log(x+iy)$. If we put n = 0, in the general value, we get the principal value of z, i.e., $\log_e \omega$.
2. $Log(-N) = \pi i + \log(N)$ Where N is positive.
3. $Log(\alpha + i\beta) = 2n\pi i + i\tan^{-1}\frac{\beta}{\alpha} + \log\sqrt{\alpha^2 + \beta^2}$

**SUMMATION OF SERIES:- C + IS METHOD**

This method is applied in finding out the sums of the series of the form

$$a_0 \cos\alpha + a_1\cos(\alpha+\beta) + a_2\cos(\alpha+2\beta) + .........$$

And $a_0 \sin\alpha + a_1\sin(\alpha+\beta) + a_2\sin(\alpha+2\beta) + .........$

Only when the sum of the series

$a_0 + a_1 x + a_2 x^2 + .........$ is known.

* $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + .....\infty$ * $e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - .....\infty$

$$* \ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\ldots\ldots\ldots\infty \qquad * \ \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\ldots\ldots\infty$$

$$* \ \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\ldots\infty \qquad * \ \log(1-x) = -\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \ldots\ldots\ldots\infty\right)$$

$$* \ \tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots\ldots\ldots\infty$$

$$* \ (1+x)^{-n} = 1 - nx + \frac{n(n+1)}{2!}x^2 - \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots\ldots\infty$$

$$* \ (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \ldots\ldots\ldots\infty$$

$$* \ (1+x)^{n} = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \ldots\ldots\ldots\infty$$

$$* \ a + ar + ar^2 + \ldots\ldots\infty = \frac{a}{1-r}\,(Infinite\ \ G.P.)$$

$$* \ a + ar + ar^2 + \ldots\ldots n = \frac{a(1-r^n)}{1-r}\,(\ finite\ \ G.P.)$$

**SHORT ANSWER TYPE QUESTIONS (2 Marks)**

1. Write DE-MOIVRE'S Theorem.2.What is C + is method, explain.
2. Write about logarithmic, exponential, hyperbolic and circular function.

**LONG ANSWER TYPE QUESTIONS (7 Marks)**

3. If $2\cos\theta = x + \frac{1}{x}$, prove that (i) $2\cos r\theta = x^r + \frac{1}{x^r}$ (ii) $\frac{x^{2n}+1}{x^{2n-1}+x} = \frac{\cos n\theta}{\cos(n-1)\theta}$
4. Show that

   (i) $(1+\sin\theta + i\cos\theta)^n + (1+\sin\theta - i\cos\theta)^n = 2^{n+1}\cos^n(\frac{\pi}{4}-\frac{\theta}{2})\cos(\frac{n\pi}{4}-\frac{n\theta}{2})$

   (ii) $\left[\frac{(1+\sin\theta+i\cos\theta)}{(1+\sin\theta-i\cos\theta)}\right]^n = c\,os(\frac{n\pi}{2}-n\alpha)+i\sin(\frac{n\pi}{2}-n\alpha)$
5. If $u = \log\left[\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right]$, prove that (i) $\tanh\frac{u}{2} = \tan\frac{\theta}{2}$. (ii) $\theta = -i\log\tan\left(\frac{\pi}{4}+\frac{iu}{2}\right)$
6. Given, $(1+x)^n = p_0 + p_1x + p_2x^2 + \ldots\ldots\ldots$

   Where n is an integer, show that:

   (i) $p_0 - p_2 + p_4 \ldots\ldots\ldots = 2^{n/2}\cos\frac{n\pi}{4}$, (ii) $p_1 - p_3 + p_5 \ldots\ldots\ldots = 2^{n/2}\sin\frac{n\pi}{4}$.
7. Prove that the $n^{th}$ roots of unity form a geometric progression. Also show that the sum of these n roots is 0 and there product is $(-1)^{n-1}$.
8. Reduce $\tan^{-1}(\cos\theta + i\sin\theta)$ to the form $A+iB$, hence show that

   $$\tan^{-1}(i\theta) = \frac{n\pi}{2} + \frac{\pi}{2} - \left(\frac{i}{2}\right)\log\left[\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]$$
9. Sum the series :

(a) $\sin\alpha.\cos\alpha+\sin^2\alpha.\cos2\alpha+\sin^3\alpha.\cos3\alpha+\text{----------}\infty$ (b) $1+x.\cos\theta+x^2.\cos2\theta+\text{------}+x^{n-1}\cos(n-1)\theta.$

(c) $1+\frac{1}{2}\cos2\theta-\left(\frac{1.3}{2.4}\right)\cos4\theta+\left(\frac{1.3.5}{2.4.6}\right)\cos6\theta-\text{.......}\infty.$ (d) $\cos\theta+\sin\theta.\cos2\theta+\left(\frac{\sin\theta}{1.2}\right)\cos3\theta+\text{.....}\infty.$

(e) $1+\frac{\cos\alpha}{\cos\alpha}+\frac{\cos2\alpha}{2!\cos^2\alpha}+\frac{\cos3\alpha}{3!\cos^3\alpha}\text{..........}\infty$ (f) $\sin\alpha-\frac{\sin(\alpha+2\beta)}{2!}+\frac{\sin(\alpha+4\beta)}{4!}+\text{..........}\infty$

(g) $1-\frac{1}{2}\cos\theta+\frac{1.3}{2.4}\cos2\theta-\frac{1.3.5}{2.4.6}\cos3\theta+\text{..........}\infty$ (h)i) $\cos\theta-\frac{1}{2}\cos2\theta+\frac{1}{3}\cos3\theta+\text{..........}\infty$

(i) $\sin\theta-\frac{1}{2}\sin2\theta+\frac{1}{3}\sin3\theta+\text{..........}\infty$

10. If $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0$, prove that

i) $\cos3\alpha+\cos3\beta+\cos3\gamma=3\cos(\alpha+\beta+\gamma)$ ii) $\sin3\alpha+\sin3\beta+\sin3\gamma=3\sin(\alpha+\beta+\gamma)$

iii) $\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)=0$ iv) $\sin(\beta+\gamma)+\sin(\gamma+\alpha)+\sin(\alpha+\beta)=0$

v) $\cos2\alpha+\cos2\beta+\cos2\gamma=0$

11. If $x+\frac{1}{x}=2\cos\theta,\quad y+\frac{1}{y}=2\cos\phi$, prove that one of the values of

i) $\frac{x^m}{y^n}+\frac{y^n}{x^m}$ is $2\cos(m\theta-n\phi)$ ii) $x^my^n+\frac{1}{x^my^n}$ is $2\cos(m\theta+n\phi)$

12. If $(a_1+ib_1)(a_2+ib_2)\text{........}(a_n+ib_n)=A+iB$, prove that

i) $(a_1^2+b_1^2)(a_2^2+b_2^2)\text{........}(a_n^2+b_n^2)=A^2+B^2$ ii) $\tan^{-1}\frac{b_1}{a_1}+\tan^{-1}\frac{b_2}{a_2}+\text{......}+\tan^{-1}\frac{b_n}{a_n}=\tan^{-1}\frac{B}{A}$

13. Solve $x^7=1$ and prove that the sum of the n^th power of the roots is 7 or zero, according as n is or is not a multiple of 7.

14. Prove by the use of De- Moivre's Theorem that the roots of the equation $(x-1)^n=x^n$ (n being a +ve integer) are $\frac{1}{2}\left[1+i\cot\frac{r\pi}{n}\right]$, where r has the values 1, 2, ……..(n-1).

15. If $i^{\alpha+i\beta}=\alpha+i\beta$, prove that $\alpha^2+\beta^2=e^{-(4n+1)\pi\beta}$.

16. If $i^{i^{i^{\cdots\infty}}}=\alpha+i\beta$, then prove that: (a) $\tan\frac{\pi A}{2}=\frac{B}{A}$ (b) $A^2+B^2=e^{-\pi\beta}$.

17. If $(a+ib)^p=m^{x+iy}$, then prove that $\frac{y}{x}=\frac{2\tan^{-1}\left(\frac{b}{a}\right)}{\log(a^2+b^2)}$ when only principal values are considered.

18. If $\sin(A+iB)=x+iy$, prove that

(i) $\frac{x^2}{\cosh^2B}+\frac{y^2}{\sinh^2B}=1$ (ii) $x^2\operatorname{cosec}^2A-y^2\sec^2A=1$

19. If $\tan(\theta+i\phi)=\cos\alpha+i\sin\alpha=e^{i\alpha}$, prove that $\theta=\frac{n\pi}{2}+\frac{\pi}{4}$ & $\phi=\frac{1}{2}\log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)$.

20. If $\tan(\theta+i\phi)=\tan\alpha+i\sec\alpha$, show that $e^{2\phi}=\pm\cot\frac{\alpha}{2}$ and $2\theta=\left(n+\frac{1}{2}\right)\pi+\alpha$.

21. If $\tan(x+iy)=\sin(u+iv)$, prove that $\frac{\sin2x}{\sinh2y}=\frac{\tan u}{\tanh v}$.

# UNIT – II

## LINEAR DIFFERENTIAL EQUATION

**DEFINATION**:- Linear differential equation are those in which the dependent variable and its derivative occur only in the first degree and are not multiplied together.

Form: $\frac{d^n y}{dx^n}+p_1\frac{d^{n-1}y}{dx^{n-1}}+p_2\frac{d^{n-2}y}{dx^{n-2}}+.......+p_n y=x \quad ----(1)$

Where $p_1, p_2 ----p_n$ are functions of x only.

* Linear differential equation with constant co-efficient are of the form

$$\frac{d^n y}{dx^n}+k_1\frac{d^{n-1}y}{dx^{n-1}}+.......+k_n y=x \quad ----(2)$$

Where $k_1, k_2 ----k_n$ are constant.

**REMARK**:- If x = 0 , then above equation (1) and (2) are called homogeneous differential equation. Otherwise it is called non-homogenous differential equation.

**OPERATOR D**

$\frac{d}{dx}, \frac{d^2}{dx^2}, \frac{d^3}{dx^3}.............$ are called D, $D^2$, $D^3$ etc. when $\frac{d}{dx}=D$

$$\Rightarrow D^n+p_1D^{n-1}y+p_2D^{n-2}y+.......p_{ny}=X$$

$$\Rightarrow [f(D)]y=X$$

**AUXILIARY EQUATION**

Put D = m, in L. H. S. part of equation (1), we get equation in the form f (m) = 0, which is called A. E. on solving A. E., we get roots of m which form complimentary function (C.F.).

| S. No. | Roots | C. F. |
|---|---|---|
| 1. | $m_1, m_2, m_3.........$ | $C_1e^{m_1x}+C_2e^{m_2x}+C_3e^{m_3x}+..........$ |
| 2. | $m_1 = m_2, m_3.........$ | $(C_1+C_2x)e^{m_1x}+C_3e^{m_3x}+..........$ |
| 3. | $m_1 = m_2 = m_3, m_4$ | $(C_1+C_2x+C_3x^2)e^{m_1x}+C_4e^{m_4x}+..........$ |
| 4. | $\alpha \pm i\beta$ | $e^{\alpha x}[C_1\cos\beta x+C_2\sin\beta x]+..........$ |
| 5. | $(\alpha \pm i\beta)^2$ (Repeated imaginary roots) | $e^{\alpha x}[(C_1+C_2x)\cos\beta x+(C_3+C_4x)\sin\beta x]$ |
| 6. | $\alpha \pm \sqrt{\beta}$ | $e^{\alpha x}[C_1\cosh\beta x+C_2\sinh\beta x]$ |

**RULES TO FIND OUT P. I.**

Particular integral depends of R. H. S. part of equation

**Case I.** When X = $e^{ax}$

$$P.I. = \frac{1}{f(D)} X = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(a)} e^{ax} \qquad Here\ f(a) \neq 0$$

Now, if f(a) = 0 Differentiate f(D) w. r. t. D.

$P.I. = x \frac{1}{f'(a)} e^{ax}$ and so on till the denominator becomes constant.

**Case – II.** When $W = \sin ax \quad or \quad \cos ax$

$$P.I. = \frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$$

$$f(-a^2) \neq 0$$

If $f(-a^2) = 0$, then

$$P.I. = x \frac{1}{f'(D^2)} \sin ax$$

**Remark:**

1. Use the formula:- $\cos^2 x = \frac{1 + \cos 2x}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}$

and $\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}$

2. Never put the value in place of D.

**Case – III.** When $X = x^m$ (m is positive integer)

$$P.I. = \frac{1}{f(D)} x^m = \left[ f(D) \right]^{-1} x^m$$

Use following formula using binomial distribution;

| | |
|---|---|
| $(1+D)^{-1}$ | $1 - D + D^2 - D^3 + ------\infty$ |
| $(1-D)^{-1}$ | $1 + D + D^2 + D^3 + ------\infty$ |
| $(1+D)^{-2}$ | $1 - 2D + 3D^2 - 4D^3 + ------\infty$ |
| $(1-D)^{-2}$ | $1 + 2D + 3D^2 + 4D^3 + ------\infty$ |

**Case – IV.** When $R.H.S. = e^{ax} \chi$

$$P.I. = \frac{1}{f(D)} e^{ax} \chi$$

$$= e^{ax} \frac{1}{f(D+a)} \chi$$

$$P.I. \quad X = e^{-ax} \chi$$

$$P.I. \quad = \frac{1}{f(D)} e^{-ax} \chi$$

$$= e^{-ax} \frac{1}{f(D-a)} \chi$$

**Case – V.** $\frac{1}{D-a} X = e^{ax} \int X e^{-ax} dx$

$$\frac{1}{D+a} X = e^{-ax} \int X e^{ax} dx$$

**Case – VI.** When R.H.S. = $x \sin x \quad or \quad x \cos x$

$$P.I. = x \frac{1}{f(D)} \sin x - \frac{f'(D)}{(f(D))^2} \sin x$$

Where $[f(D) \neq 0]$

## VARIATION OF PARAMETER METHOD

1) If $C.F. = c_1 y_1 + c_2 y_2$
   Then we calculate
   $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$ (wronskian) $(\neq 0)$
2) $P.I. = -y_1 \int \frac{y_2 x}{W} dx + y_2 \int \frac{y_1 x}{W} dx$

## CAUCHY HOMOGENEOUS LINEAR EQUATION

**AN EQUATION OF THE FORM**

$$x^n \frac{d^n y}{dx^n} + k_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + k_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + ....... + k_n y = X$$

Where X is a function of x, is called Cauchy Homogeneous L. D. E.

**Procedure:-**

Write $x \frac{dy}{dx} = Dy$ , $x^2 \frac{d^2 y}{dx^2} = D(D-1)y$ ,

$x^3 \frac{d^3 y}{dx^3} = D(D-1)(D-2)y$ and so on.

## LEGENDRE HOMOGENEOUS EQUATION

$$(ax+b)^n \frac{d^n y}{dx^n} + k_1 (ax+b)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ----- + k_n y = f(x)$$

Where, $k_1, k_2, ------, k_n$ are constants is called Legendre Linear Equation

Here put

$$(ax+b)\frac{dy}{dx} = aDy$$

$$(ax+b)^2\frac{d^2y}{dx^2} = a^2D(D-1)y$$

and so on.

### SHORT ANSWER TYPE QUESTIONS (2Marks)

1. Define Linear Differential Equation.
2. Explain method of variation of parameter.
3. Define Cauchy and Legendre Linear Equation.

### LONG ANSWER TYPE QUESTIONS (7 Marks)

4. Solve $(D^2+2)y = x^2e^{3x} + e^x\cos 2x.$
5. Solve $x^2y''+xy'+y = \log x.\sin(\log x).$
6. Solve $(D-2)^2y = 8(e^{2x}+\sin 2x + x^2).$
7. Solve $\frac{d^2y}{dx^2} - 4y = x\sinh x.$
8. Solve $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x\sin x$
9. Solve $(D^2-4D+4)y = 8x^2e^{2x}\sin 2x.$
10. Solve $\frac{d^2y}{dx^2} - 4y = \cosh h(2x-1) + 3^x.$
11. Solve $(D^4+D^2+1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x.$
12. Solve $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = e^{e^x}$
13. Using the method of variation of parameters, solve

    (I) $\frac{d^2y}{dx^2} + 4y = \tan 2x.$ (II) $y'' - 2y' + y = e^x\log x.$

    (III) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = \frac{e^x}{1+e^x}$ (IV) $\frac{d^2y}{dx^2} + y = \frac{1}{1+\sin x}$ (V) $(D^2+1)y = x\sin x.$
14. Solve $x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}.$
15. Solve $x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x.$

16. The radial displacement u in a rotating disc at a distance r from the axis is given by $r^2 \frac{d^2u}{dr^2} + r\frac{du}{dr} - u + kr^3 = 0$, where k is a constant. Solve the equation under the condition u = 0 when r = 0, u = 0 when r = a.

17. Solve $x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log x)$.

18. Solve $x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10\left(x + \frac{1}{x}\right)$.

19. Solve $(3x+2)^2 \frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2 + 4x + 1$.

20. Solve $(2x-1)^2 \frac{d^2y}{dx^2} + (2x-1)\frac{dy}{dx} - 2y = 8x^2 - 2x + 3$.

21. Solve the simultaneous equations: $\frac{dx}{dt} + 2y + \sin t = 0, \quad \frac{dy}{dt} - 2x - \cos t = 0$ given that x = 0 and y = 1 when t = 0.

22. Solve the simultaneous equations: $\frac{dx}{dt} = 2y, \quad \frac{dy}{dt} = 2z, \quad \frac{dz}{dt} = 2x$.

23. Solve the simultaneous equations: $\frac{dx}{dt} + y = \sin t, \quad \frac{dy}{dt} + x = \cos t$: given that x = 2 and y = 0 when t = 0

24. Solve the simultaneous equations: $\frac{dx}{dt} + 2y = e^t, \quad \frac{dy}{dt} - 2x = e^{-t}$

25. Solve the simultaneous equations: $t\frac{dx}{dt} + y = 0, \quad t\frac{dy}{dt} + x = 0$ given x(1) = 1, y(-1) = 0.

26. Solve the simultaneous equations: $\frac{dx}{dt} + \frac{dy}{dt} + 3x = \sin t, \quad \frac{dx}{dt} + y = \cos t$ .

27. Solve the simultaneous equations: $\frac{d^2x}{dt^2} + y = \sin t, \quad \frac{d^2y}{dt^2} + x = \cos t$ .

28. Solve the simultaneous equations: $\frac{dx}{dt} = ny - mz, \quad \frac{dy}{dt} = lz - nx, \quad \frac{dz}{dt} = mx - ly$.

## UNIT – V
## THEORY OF EQUATIONS

**POLYNOMIALS:**

$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + - - - - - - a_n$, where n is a +ve integer and $a_0, a_1, - - - - - - a_n$ are constants, is called a polynomial of n^th degree if $a_0 \neq 0$.

If $f(x) = 0$, it is called algebraic or polynomial equation.

If $f(x)$ contain $e^x, \log x, \sin x, \cos x$, it is called Transcendental equation.

**PROPERTIES OF EQUATIONS**

I) Every equation has a Root, Real or imaginary.

II) Every polynomial equation of the $n^{th}$ degree has n and only n Roots.

III) Imaginary Roots occur in conjugate pair. If $\alpha + i\beta$ is a Root of equation, then $\alpha - i\beta$ is also a root of equation.

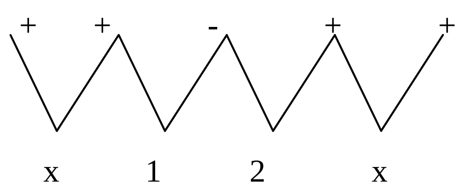
IV) Irrational Root occurs in pain. If $\alpha + \sqrt{\beta}$ is a Root of equation, Then $\alpha - \sqrt{\beta}$ is also a Root.

V) If $x = \alpha$ is a Root of equation, then $(x - \alpha)$ is a factor of it.

VI) **Intermediate valve property**:- if f(a) and f(b) have different signs, then f(x) = 0 has at least one root lies between x = a and x = b.

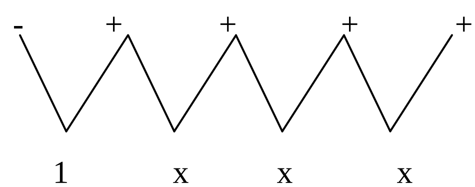
VII) **Descarte's Rule of signs**:- The equation f(x) = 0 cannot have more positive roots than the changes of signs in f(x); and more negative Roots then the change s of sign in f(-x).

e.g. $$f(x) = x^5 + 3x^4 - 3x^3 + 4x^2 + 5$$




2 Changes:- Two positive Roots

$$f(-x) = -x^5 + 3x^4 + 3x^3 + 4x^2 + 5$$




One change, one negative Root

<u>Hence no. of imaginary Roots</u>

= 5 – (2 + 1) = 2

VIII) **Synthetic Division Method**: This method is quite useful to find out Roots of given equation, if one Root is known

e.g. $$x^5 + 4x^4 + x^3 + x^2 + 4x + 1 = 0$$

here x = - 1 is always a Root.

| | | | | | | |
|---|---|---|---|---|---|---|
| -1 | 1 | 4 | 1 | 1 | 4 | 1 |
| | - | -1 | -3 | 2 | -3 | -1 |
| | 1 | 3 | -2 | 3 | 1 | 0 |

New equation is $x^4 + 3x^3 - 2x^2 + 3x + 1 = 0$

Synthetic division method also used to diminish the equation.

| S. NO. | ROOTS | TERM |
|---|---|---|
| 1. | Three Roots in A. P. | a – d, a , a + d |
| 2. | For Roots in A. P. | a – 3d, a - d , a + d, a + 3d |
| 3. | Three Roots in G. P. | a / r, a, ar |
| 4. | For Roots in G. P. | $a / r^3$, a / r, ar, $ar^3$ |

Remark:

1. If a, b, c are in A. P.

then $b = \frac{a+c}{2}$

2. If a, b, c are in G. P.

then $b^2 = ac$

3. If roots are in H. P.,

$x = \frac{1}{x}$ in the equation and solve as A. P. or a, b, c are in H. P. $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$

4. Synthet

**RELATION BETWEEN ROOTS AND CO-EFFICIENT**

**I) QUADRATIC EQUATIONS**

$$ax^2 + bx + c = 0$$

$$\alpha + \beta = \frac{-b}{a} \text{ (Sum of Roots)}$$

$$\alpha\beta = \frac{c}{a} \text{ (Product of Roots)}$$

$\therefore$ New equation is

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$

**II) CUBIC EQUATIONS**

$$ax^3 + bx^2 + cx + d = 0$$

Let its Roots are $\alpha, \quad \beta, \quad \gamma$

$$\alpha + \beta + \gamma = \sum \alpha = \frac{-b}{a}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \sum \alpha\beta = \frac{c}{a}$$

$$\alpha\beta\gamma = \frac{-d}{a}$$

**III) QUADRATIC EQUATIONS**

$$ax^4 + bx^3 + cx^2 + dx + e = 0$$

Let its Roots are $\alpha, \quad \beta, \quad \gamma, \quad \delta$

$$\alpha + \beta + \gamma + \delta = \frac{-b}{a} = \sum \alpha$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = \sum \alpha\beta = \frac{c}{a}$$

$$\alpha\beta(\gamma+\delta)+\gamma\delta(\alpha+\beta)=\frac{-d}{a}=\sum\alpha\beta\gamma$$

$$\alpha\beta\gamma\delta=\frac{e}{a}$$

**RELATED FORMULAE**

i) $\sum\alpha^2=\alpha^2+\beta^2+\gamma^2=\left(\sum\alpha\right)^2-2\left(\sum\alpha\beta\right)$

ii) $\sum\alpha^3=\alpha^3+\beta^3+\gamma^3=\left(\sum\alpha\right)^3-3\left(\sum\alpha\right)\left(\sum\alpha\beta\right)+3\alpha\beta\gamma$

But if $\alpha+\beta+\gamma=0$

Then $\alpha^3+\beta^3+\gamma^3=3\alpha\beta\gamma$

iii) $\sum\alpha^2\beta=\sum\alpha\sum\alpha\beta-3\alpha\beta\gamma$

iv) $\sum\alpha^2\beta^2=\left(\sum\alpha\beta\right)^2-2\alpha\beta\gamma\left(\sum\alpha\right)$

v) $\sum\alpha^4=\left(\sum\alpha^2\right)^2-2\sum\alpha^2\beta^2$

vi) $\sum\alpha^2\beta\gamma=\alpha^2\beta\gamma+\beta^2\alpha\gamma+\gamma^2\alpha\beta=\alpha\beta\gamma(\alpha+\beta+\gamma)$

**TRANSFORMATION OF EQUATIONS**

I) To find an equations of whose roots are m time the roots of equation.

Multiply the II[nd] term by m, third term by $m^2$ and so on.

e. g. $x^3+3x^2+4x+1=0$ (Roots are 3 times the of an equation)

$$x^3+(3X3)x^2+(9X4)x+1X27=0$$

$$x^3+9x^2+36x+27=0$$

II) To find an equation whose Roots are with opposite sign to those of the given equation.

Change the signs of every alternative term of the given equation beginning with the second.

e. g. $x^3+x^2+x+1=0$

New equation is e. g. $x^3-x^2+x-1=0$

III) To find an equation whose Roots are the reciprocals of the Roots of given equation.

Change $x$ *to* $1/x$.

Equation $x^3+3x^2+x+3=0$

$$\therefore\left(\frac{1}{x}\right)^3+3\left(\frac{1}{x}\right)^2+\left(\frac{1}{x}\right)+3=0$$

IV) To diminish the Roots of an equation f(x) = 0 by **h** by synthetic division method. Divide f(x) by (x – h) successively.

To increase the Roots by h, we take h **negative**.

V) RECIPROCAL EQUATIONS:

If an equation does not change an changing x into 1 / x, it is called Reciprocal equations.

| S. NO. | NATURE OF EQUATIONS | ROOTS |
|---|---|---|
| 1. | Degree – odd, co – efficient of terms equidistant form beginning and end are equal. | - 1 |
| 2. | Degree – odd, co – efficient equal but opposite in sign | + 1 |
| 3. | Degree – even, co – efficient equal but opposite in sign | + 1, - 1. |

**CARDAN METHOD**

To solve cubic equation, we use Cardan method. These are of two tsypes:

**I) In which term of $x^2$ absent**

Equation $ax^3+bx+c=0 ---------(1)$

Here we put $x=u+v$

and $x^3=(u+v)^3$

$$x^3=u^3+v^3+3uv(u+v)$$

$$x^3=u^3+v^3+3uvx$$

$$x^3-3uvx-(u^3+v^3)=0 ---------(2)$$

Compare (1) and (2), we get

$$-3uv=b \Rightarrow uv=\frac{-b}{3} \Rightarrow (uv)^3=\left(\frac{-b}{3}\right)^3=\frac{-b^3}{27}$$

$$-(u^3+v^3)=c \Rightarrow u^3+v^3=-c$$

**II) In which term of $x^2$ present:**

$$ax^3+bx^2+cx+d=0$$

Here, we get removed the $x^2$ term by diminishing the Roots by

$h=\frac{-b}{na}$ (Where n = degree = 3)

**SHORT ANSWER TYPE QUESTIONS (2 Marks)**

1. Explain Discarte's Rule of sign.
2. Write intermediate value property.
3. Define reciprocal equation.
4. Find the equation where roots are 3 times the roots of equation $x^4-3x^3+2x^2-7x+1=0$
5. Find the Reciprocal equation of the equation $x^3+2x^2+7x-2=0$.

**LONG ANSWER TYPE QUESTIONS (7 Marks)**

1. The equation $x^4-4x^3+ax^2+b=0$ has two pairs of equal roots. Find the values of **a** and **b**.
2. O, A, B, C are four points on a straight line such that the distances of A, B, C from O are the roots of equation $ax^3+3bx^2+3cx+d=0$. If B is the middle point of AC, show that $a^2d-3abc+2b^3=0$.
3. Solve the equation $x^4+5x^3-30x^2-40x+64=0$ whose roots are in G.P.

4. If $\alpha, \beta, \gamma$ are the roots of equation $x^3 + qx + r = 0$, find the equation whose roots are $\frac{\beta}{\gamma} + \frac{\gamma}{\beta}, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma}, \frac{\beta}{\alpha} + \frac{\alpha}{\beta}$.

5. Find the condition such that the equation $x^3 + px^2 + qx + r = 0$ has roots $\alpha, \beta$ which satisfy $\alpha\beta + 1$.

6. If $\alpha, \beta, \gamma$ are the roots of equation $x^3 + qx + r = 0$, find the equation whose roots are $(\alpha - \beta)^2, (\beta - \gamma)^2, (\gamma - \alpha)^2$.

7. Solve the reciprocal equations

(i) $6x^6 - 25x^5 + 31x^4 - 31x^2 + 25x - 6 = 0$ (ii) $8x^5 - 22x^4 - 55x^3 + 55x^2 + 22x - 8 = 0$

8. Solve the equation $x^4 - 2x^3 + 4x^2 + 6x - 21 = 0$, given that the sum of two of its roots is zero.

9. If $\alpha, \beta, \gamma$ be the roots of the equation $x^3 + px + q = 0$, find the value of

(a) $\sum \alpha^2 \beta$ (b) $\sum \alpha^4$ (c) $\sum \alpha^3 \beta$

10. Solve the equations (i) $x^4 + 2x^3 - 21x^2 - 22x + 40 = 0$ whose roots are in A. P.

(ii) $x^4 + 5x^3 - 30x^2 + 40x + 64 = 0$ whose roots are in G. P.

11. If $\alpha, \beta, \gamma$ are the roots of equation $x^3 + 4x - 3 = 0$, find the value of

(i) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ (ii) $\alpha^{-2} + \beta^{-2} + \gamma^{-2}$

12. If $\alpha, \beta, \gamma$ are the roots of equation $x^3 + px + q = 0$, show that

(i) $\alpha^5 + \beta^5 + \gamma^5 = 5\alpha\beta\gamma\left(\beta\alpha + \gamma\alpha + \alpha\beta\right)$ (ii) $3\sum\alpha^2 \sum\alpha^5 = 5\sum\alpha^3 \sum\alpha^4$

13. Transform the equation $x^3 - 6x^2 + 5x + 8 = 0$ into another in which the second term is missing. Hence find the equation of its squared differences.

14. If $\alpha, \beta, \gamma$ are the roots of equation $x^3 + mx + n = 0$, from the equation whose roots are

(a) $\alpha + \beta - \gamma,\ \beta + \gamma - \alpha,\ \gamma + \alpha - \beta$. (b) $\frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}, \frac{\alpha\beta}{\gamma}$ (c) $\frac{1}{\beta} + \frac{1}{\gamma}, \frac{1}{\gamma} + \frac{1}{\alpha}, \frac{1}{\alpha} + \frac{1}{\beta}$

15. Show that the equation $x^4 - 10x^3 + 23x^2 - 6x - 15 = 0$ can be transformed into reciprocal equation by diminishing the roots by 2. Hence solve the equation.

16. By suitable transformation, reduce the equation $x^4 + 16x^3 + 83x^2 + 152x + 84 = 0$ to an equation in which term in $x^3$ is absent and hence solve it.

17. Solve the cubic by Cardan's method.

(a) $28x^3 - 9x^2 + 1 = 0$ (b) $x^3 + x^2 - 16x + 20 = 0$ (c) $x^3 - 3x^2 + 3 = 0$.

(d) $9x^3 + 6x^2 - 1 = 0$ (e) $x^3 - 3x + 1 = 0$ (f) $27x^3 + 54x^2 + 198x - 73 = 0$

18. Solve the equation by Ferrari's method

(a) $x^4-12x^3+41x^2-18x-72=0$ (b) $x^4-2x^3-5x^2+10x-3=0$ (c) $x^4-10x^3+35x^2-50x+24=0$

(d) $x^4-10x^2-20x-16=0$

19. If the roots of the equation $x^3+3px^2+3qx+r=0$ are in H. P., then prove that $2q^3=r(3pq-r)$.

20. If $\alpha,\beta,\gamma$ are the roots of the equation $x^3+ax^2+bx+c=0$ than find

(a) $\sum\alpha^2$ (b) $\sum\frac{1}{\alpha}$ (c) $\sum\alpha^2\beta$ (d) $\sum\alpha^3$ (e) $\sum\beta^2\alpha^2$

(f) $\sum(\beta+\gamma)(\gamma+\alpha)(\alpha+\beta)$

21. If the roots of the equation $x^3+3px^2+3qx+r=0$ are in G. P., then prove that $p^3r=q^3$.

22. Find the condition, when the roots of the equation $x^3+3px^2+3qx+r=0$ are in H. P.

23. If the roots of the equation $x^3-px^2+qx-r=0$ are in H. P., then prove that $27r^2-9pqr+2q^3=0$.

24. If $\alpha,\beta,\gamma$ are the roots of equation $x^3+qx+r=0$, from the equation whose roots are

(a) $\frac{\beta+\gamma}{\alpha^2},\frac{\gamma+\alpha}{\beta^2},\frac{\alpha+\beta}{\gamma^2}$. (b) $\frac{\beta}{\gamma}+\frac{\gamma}{\beta},\frac{\gamma}{\alpha}+\frac{\alpha}{\gamma},\frac{\alpha}{\beta}+\frac{\beta}{\alpha}$ (c) $\frac{\beta^2+\gamma^2}{\alpha^2},\frac{\gamma^2+\alpha^2}{\beta^2},\frac{\alpha^2+\beta^2}{\gamma^2}$